



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE ANALYST.

VOL. IV.

SEPTEMBER, 1877.

No. 5.

SHORT METHOD OF ELLIPTIC FUNCTIONS.

BY LEVI W. MEECH, A. M., HARTFORD, CONN.

THE system of Elliptic Functions is closely connected with analytical geometry and trigonometry, and may be said to rank next in importance. The earlier advances of Fagnani, Euler, Landen and Lagrange were expanded in a systematic treatise by Legendre, whose well directed labor of forty years is also allied with the brilliant discoveries of Abel and Jacobi, and many others. Their researches have greatly facilitated the solution of numerous problems in Mechanics, Astronomy, Geometry, and the theory of Heat and Light.

The present compendium has been arranged as free from complexity as possible. Especially in the seventh Section, the unexpected discovery of a new scale of amplitudes adapted equally to successive or to simultaneous quadrature, happily attains the object of the theorems of Jacobi and Ivory, without the previous long and intricate demonstration. And the common origin of all known scales of moduli is traced to a simple integral in the next Section.

- | | | |
|---------|-------|---|
| SECTION | I. | Change from Algebraic to Elliptic Functions. |
| " | II. | Connection with the Spherical Triangle. |
| " | III. | Increasing Scales of Moduli. |
| " | IV. | Decreasing Scales of Moduli. |
| " | V. | Integration by series. Table of Quadrants. |
| " | VI. | Inverse Method of Integration. Theta Functions. |
| " | VII. | Successive and Simultaneous Quadratures. |
| " | VIII. | Quadrantal Quadrature. |
| " | IX. | Elliptic Functions of the Second and Third Order. |

DEFINITIONS. Elliptic Functions, or Transcendents, were classed by Legendre into three species. And the complete periphery of each is a closed curve having four similar quadrants. The Function of the first species is denoted by F . The second, which is an arc of the common ellipse, is denoted by E ; and the third by Π . Thus,

$$F = F(e, \theta) = \int_0^\theta \frac{d\theta}{\sqrt{[1 - e^2 \sin^2 \theta]}}.$$

$$E = E(e, \theta) = \int_0^\theta \sqrt{[1 - e^2 \sin^2 \theta]} d\theta.$$

$$\Pi = \Pi(e, n, \theta) = \int_0^\theta \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{[1 - e^2 \sin^2 \theta]}}.$$

To abbreviate, let $A = \sqrt{1 - e^2 \sin^2 \theta}$; $b = \sqrt{1 - e^2}$; $F(e, \frac{1}{2}\pi) = A \cdot \frac{1}{2}\pi$.

The variable arc θ is named the *amplitude* of the function. And e , being a fraction not greater than 1, is called the *modulus*, or, *eccentricity*; while b is termed the complementary modulus. Also n is termed the *parameter*. If e is denoted by $\sin \varphi$, and b by $\cos \varphi$, then φ is sometimes called the angle of the modulus.

Examples of F. From Legendre's *Fonctions Elliptiques*, Vol. 2, Table 9:

e .	$\theta = 25^\circ$.	$\theta = 45^\circ$.	$\theta = 70^\circ$.	$\theta = 90^\circ$.
Sin 25°	0.43874 7923	0.79870 514	1.26548 460	1.64899 522
" 45°	0.44328 23288	0.82601 7876	1.36971 9477	1.85407 4677
" 70°	0.44903 50293	0.86652 9957	1.59590 6244	2.50455 0079
" 90°	0.45087 533	0.88137 359	1.73541 516	∞

Values of E.

" 25°	0.44339 4028	0.77247 109	1.18039 569	1.49811 493
" 45°	0.42957 52476	0.74818 6504	1.09900 8292	1.35064 3881
" 70°	0.42242 64952	0.71714 7672	0.98297 5826	1.11837 774
" 90°	0.42261 826	0.70710 6781	0.93969 2621	1.00000 0000

I. CHANGE FROM ALGEBRAIC TO ELLIPTIC FUNCTIONS. This is generally required by the occurrence of a radical denominator of the *third* or *fourth* degree. The first step is to resolve into two quadratic factors; thus the denominator $R = \sqrt{[A + Bx + Cx^2 + Dx^3 + Ex^4]}$

$$= \sqrt{[(a + bx + cx^2)(a' + b'x + c'x^2)]}.$$

The second step is to eliminate the odd powers of the variable. For this object, one method consists in changing to y^2 , after assuming

$$a + bx + cx^2 = (a' + b'y + c'y^2)y^2.$$

Another simpler assumption is, $x = (p + qy) \div (1 + y)$. Substituting into the preceding factors, and equating each coefficient of y to zero,

$$a + \frac{1}{2}b(p+q) + c pq = 0,$$

$$a' + \frac{1}{2}b'(p+q) + c' pq = 0.$$

These give by elimination the sum $p+q$, and the product pq ; then p, q are roots of the quadratic $u^2 - (p+q)u + pq = 0$. Should these roots come out imaginary, Legendre proves there are two other combinations of the binomial factors $(x-r)$ etc., of R , which would give real roots for p, q . The two factors of R thus take the required form $f+gy^2$.

1. When the factors of R , or $(a^2 + 2ab \cos s.x^2 + b^2x^4)^{\frac{1}{2}}$, are imaginary.

Let $x = \sqrt{\left(\frac{a}{b}\right) \tan \frac{1}{2}\theta}$, $e = \sin \frac{1}{2}s$; then $\frac{dx}{R} = \frac{1}{2\sqrt{(ab)}} \cdot \frac{d\theta}{\sqrt{(1-e^2 \sin^2 \theta)}}$.

2. When $R = \sqrt{[a^2(1+b^2x^2)(1-e^2x^2)]}$. Let $x = (1 \div c) \cos \theta$;

$$\frac{b^2}{b^2 + c^2} = e^2; \text{ which gives } \frac{dx}{R} = \frac{e}{ab} \cdot \frac{-d\theta}{\sqrt{[1-e^2 \sin^2 \theta']}}.$$

To make the result positive, let $\cot \theta = \sqrt{(1-e^2)} \tan \theta'$; then

$$\frac{dx}{R} = \frac{e}{ab} \cdot \frac{d\theta'}{\sqrt{[1-e^2 \sin^2 \theta']}}.$$

3. When $R^2 = (1+a^2x^2)(x^2-b^2)$. Let $x = \frac{b}{\cos \theta}$; $\frac{1}{1+a^2b^2} = e^2$; $\frac{dx}{R}$

$$= \frac{ed\theta}{\sqrt{[1-e^2 \sin^2 \theta]}}.$$

4. When $R^2 = (1+a^2x^2)(1+b^2x^2)$. Suppose $a > b$; $x = \frac{\tan \theta}{a}$; $\frac{a^2-b^2}{a^2} = e^2$.

5. When $R^2 = (1-a^2x^2)(1-b^2x^2)$; $a > b$; let $ax = \sin \theta$; $b \div a = e$; then $\frac{dx}{R} = \frac{1}{a} \cdot \frac{d\theta}{\sqrt{[1-e^2 \sin^2 \theta]}}$. The radical will be imaginary from $x = \frac{1}{a}$ to $\frac{1}{b}$,

beyond which it becomes real to $x = \infty$. For these real values, $R^2 = (a^2x^2-1)(b^2x^2-1)$. Let $bx = \frac{1}{\sin \theta}$; $\frac{b}{a} = e$; $\frac{dx}{R} = \frac{-d\theta}{a \sqrt{[1-e^2 \sin^2 \theta]}}$.

6. When $R^2 = (a^2-x^2)(x^2-b^2)$; x lies between a and b . If $a > b$, let $e^2 = \frac{a^2-b^2}{a^2}$, and $x^2 = \frac{b^2}{1-e^2 \sin^2 \theta}$; $\frac{dx}{R} = \frac{1}{a} \cdot \frac{d\theta}{\sqrt{[1-e^2 \sin^2 \theta]}}$.

See Legendre's treatise, Vol. I, p. 4, . . . p. 252.

7. When $R^2 = a+bx^2+cx^4+dx^6$; let $x^2 = z$, etc.

8. When $R^4 = a+bx^2+cx^4$; let $R = z$ or xy , etc.

9. When $R^3 = a+bx+cx^2+dx^3$; make $R = x.d^{\frac{1}{3}}+z$, etc.

10. When $R^2 = (a+b \sin^2 \theta)(c+d \sin^2 \theta)$; change first from $\sin \theta$ to $\tan \theta$.

11. When $R^2 = A+B \cos \theta + C \sin \theta + D \cos^2 \theta + E \sin \theta \cos \theta + F \sin^2 \theta$;

make $\tan \frac{1}{2}\theta = z$, or $\sin \theta = \frac{2z}{1+z^2}$, $\cos \theta = \frac{1-z^2}{1+z^2}$, $d\theta = \frac{2dz}{1+z^2}$; etc.

OTHER INTEGRAL FORMS.

12. $R^2 = (x-a)(x-b)(x-c)$; make one factor, as $x-a = y^2$.
- 13, 14. $\int \frac{\sin^2 \theta d\theta}{A} = \frac{1}{e^2} (F-E)$; $\int \frac{\cos^2 \theta d\theta}{A} = \frac{1}{e^2} (E-b^2 F)$.
- 15, 16. $\int \frac{\tan^2 \theta d\theta}{A} = \frac{1}{b^2} (A \tan \theta - E)$; $\int \frac{d\theta}{A^3} = \frac{1}{b^2} E - \frac{e^2 \sin \theta \cos \theta}{b^2 A}$.
17. $\int \frac{\sin^2 \theta d\theta}{A^3} = \frac{1}{b^2 e^2} (E-b^2 F) - \frac{\sin \theta \cos \theta}{b^2 A}$.
18. $\int \frac{\cos^2 \theta d\theta}{A^3} = \frac{1}{e^2} (F-E) + \frac{\sin \theta \cos \theta}{A}$.
19. $\int \frac{d\theta}{A^5} = \frac{2(1+b^2)}{3b^4} E - \frac{1}{3b^2} F - \frac{e^2 \sin \theta \cos \theta}{3b^4 A} \left(2 + 2b^2 + \frac{b^2}{A^2}\right)$.
20. $\int \frac{d\theta}{A \cos^2 \theta} = \frac{1}{b^2} (A \tan \theta + b^2 F - E)$.
- 21, 22. $\int \frac{A d\theta}{\cos^2 \theta} = A \tan \theta + F - E$; $\int A \tan^2 \theta d\theta = A \tan \theta + F - 2E$.
23. $\int A \sin^2 \theta d\theta = -\frac{1}{3} A \sin \theta \cos \theta + \frac{2e^2-1}{3e^2} E + \frac{b^2}{3e^2} F$.
24. $\int A^3 d\theta = \frac{1}{3} e^2 A \sin \theta \cos \theta + \frac{2+2b^2}{3} E - \frac{b^2}{3} F$.
25. $\int \frac{d\theta}{A \cos^2 \frac{1}{2} \theta} = 2A \tan \frac{1}{2} \theta + 2F - 2E$.

II. CONNECTION WITH THE SPHERICAL TRIANGLE. In spherical trigonometry it is proved that when one side and its opposite angle, as c, C , are regarded as constant (Chauvenet's Trigonometry, p. 239), we have

$$\frac{da}{\cos A} + \frac{db}{\cos B} = 0.$$

The constancy of c, C , makes the ratio e constant in the same triangle; thus, $\sin A = e \sin a$; $\sin B = e \sin b$; $\sin C = e \sin c$;
 $\cos A = \sqrt{[1-e^2 \sin^2 a]}$, $\cos B = \sqrt{[1-e^2 \sin^2 b]}$, $\cos C = -\sqrt{[1-e^2 \sin^2 c]}$.

If c denote the longest side, C will denote the greatest angle, and the equation below requires that $\cos C$ have a sign opposite to that of $\cos A$ and $\cos B$. Now substituting the radical values in place of $\cos A$ and $\cos B$ above, and integrating,

$$\int \frac{da}{\sqrt{[1-e^2 \sin^2 a]}} + \int \frac{db}{\sqrt{[1-e^2 \sin^2 b]}} = \int C' dc.$$

For, had the three sides been regarded as variable, the differential equation quoted above, would have a third term of the form $C' dc$. As arbitrary constant here, it can contain neither a nor b . When $a=0, b=c$; when

$b = 0, a = c$; in either case the last equation defines the form of the function C' ; thus,

$$\int \frac{da}{\sqrt{[1-e^2 \sin^2 a]}} + \int \frac{db}{\sqrt{[1-e^2 \sin^2 b]}} = \int \frac{dc}{\sqrt{[1-e^2 \sin^2 c]}}.$$

Thus the function F of the first species has the remarkable property of being expressed by two similar terms, which are not separately integrable. The relation was first discovered by the celebrated Euler in 1761.

In respect to E , or the second species, we differentiate the preceding equation, clear of fractions and restore the cosines before described; giving

$$\cos B \cos C.da + \cos A \cos C.db + \cos A \cos B.dc = 0.$$

Substituting for each product of cosines, its value from the well known formula of spherical trigonometry,

$$\begin{aligned} \cos A \cos B &= \sin A \sin B \cos c - \cos C, \text{ or} \\ &= e^2 \sin a \sin b \cos c - \cos C; \end{aligned}$$

$$\begin{aligned} \text{we find } \cos A.da + \cos B.db + \cos C.dc \\ &= e^2 [\sin a \sin b \cos c.dc + \sin a \sin c \cos b.db + \sin b \sin c \cos a.da] \\ &= e^2 d(\sin a \sin b \sin c). \end{aligned}$$

Replacing the radical values of the cosines, and integrating,

$$\begin{aligned} \int \sqrt{[1-e^2 \sin^2 a]}da + \int \sqrt{[1-e^2 \sin^2 b]}db &= \int \sqrt{[1-e^2 \sin^2 c]}dc \\ &+ e^2 \sin a \sin b \sin c. \end{aligned}$$

$$\text{Or } E(a) + E(b) = E(c) + e^2 \sin a \sin b \sin c.$$

The equation of the *Ellipse* whose rectangular coordinates x, y , have their origin at the center, is

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} = 1.$$

Let θ denote an auxiliary angle; and since $\sin^2 \theta + \cos^2 \theta = 1$, we can make $y = B \sin \theta, x = A \cos \theta$; $B^2 = A^2(1-e^2)$. Then the arc

$$\sqrt{[dx^2 + dy^2]} = A\sqrt{[1-e^2 \sin^2 \theta]}d\theta.$$

The latter member, when A is unity, is identical with dE ; and the relation of the auxiliary arc θ to the co-ordinates is manifest. Again if the preceding value of y is equated to that which is found from taking the focus nearest the apsis as origin, we find by comparison that θ is the angle named in Astronomy the *eccentric anomaly*. And thus, if v denote the *true anomaly*, we have

$$\sin \theta = \frac{\sqrt{(1-e^2)}\sin v}{1+e \cos v}, \quad \cos \theta = \frac{e + \cos v}{1+e \cos v};$$

and the preceding relation of three amplitudes or eccentric anomalies is applicable to three distances on the orbit.

The equation of the *Hyperbola*, whose rectangular co-ordinates have their origin at the center, is $A^2y^2 - B^2x^2 = -A^2B^2$.

If we make $A \div \sqrt{A^2 + B^2} = e$; $y = B\sqrt{1 - e^2} \tan \theta$; where θ denotes an auxiliary angle; $x = \sqrt{A^2 \div \cos^2 \theta - A^2 e^2 \tan^2 \theta}$; then the hyperbolic arc

$$\sqrt{dx^2 + dy^2} = B\sqrt{1 - e^2} \cdot \frac{d\theta}{\cos^2 \theta \sqrt{1 - e^2 \sin^2 \theta}} = B\sqrt{1 - e^2} \cdot dH.$$

The Imaginary. Here let us change to θ' , and after integration, return to θ ; first assuming the symmetrical relation $\cos \theta \cos \theta' = 1$. Then, $\sin \theta = \sqrt{(-1) \tan \theta'}$; $\sin \theta' = \sqrt{(-1) \tan \theta}$; $d\theta = \sqrt{(-1)} (d\theta' \div \cos \theta')$;

$$dH = \frac{d\theta}{\cos^2 \theta \sqrt{1 - e^2 \sin^2 \theta}} = \frac{\sqrt{(-1)} \cos^2 \theta' d\theta'}{\sqrt{1 - b^2 \sin^2 \theta'}}.$$

Here $b^2 = 1 - e^2$. By division and integration this becomes,

$$H(e, \theta) = \frac{\sqrt{-1}}{b^2} \left[E(b, \theta') - e^2 F(b, \theta') \right].$$

Replacing the symmetrical arcs θ, θ' by a, a' , etc., adding the equations for a, b , and subtracting for c as the greatest side of the spherical triangle, the three terms of F become zero, as before shown, while those of E with accented values change signs, and are equal to the product below;

$$H(e, a) + H(e, b) - H(e, c) = \frac{-\sqrt{-1}}{b^2} \left(b^2 \sin a' \sin b' \sin c' \right) = -\tan a \tan b \tan c.$$

Taking $e^2 = \frac{1}{2}$, $c = 70^\circ$ and $b = 45^\circ$, we find by the spheric formula below, $a = 30^\circ 26'$. Then $H(c) = 3.2782$, $H(b) = 1.0617$, $H(a) = 0.6025$; and each member becomes -1.6140 , which verifies the preceding formula. It therefore merits our confidence. The abstruse subject of integration through imaginary limits has been systematized by Cauchy and others.

Among the earlier results, the following is readily proved by differentiation.

$$H = \int \frac{d\theta}{\cos^2 \theta \cdot A} = \frac{A \tan \theta - E + (1 - e^2) F}{1 - e^2}.$$

To recapitulate,—when e is less than unity, let c denote the longest of the three sides a, b, c of the spherical triangle. Then will the three sides or amplitudes, and their elliptic integrals be connected by the following equations:

$$\cos c = \cos a \cos b - \sin a \sin b \sqrt{1 - e^2 \sin^2 c},$$

$$\cos a = \cos b \cos c + \sin b \sin c \sqrt{1 - e^2 \sin^2 a},$$

$$\cos b = \cos a \cos c + \sin a \sin c \sqrt{1 - e^2 \sin^2 b}.$$

$$F(a) + F(b) = F(c).$$

$$E(a) + E(b) = E(c) + e^2 \sin a \sin b \sin c.$$

$$H(a) + H(b) = H(c) - \tan a \tan b \tan c.$$

$$H(a) + H(b) = H(c) + \frac{n}{N} \tan^{-1} \left[\frac{N \sin a \sin b \sin c}{1 + n(1 - \cos a \cos b \cos c)} \right].$$

Here $N = \sqrt[n(n+1)(n+e^2)]$; and when n is negative, the last term is to be changed to a logarithmic form. Similar summations might evidently be obtained for the various integrals at the end of Section I.

III. INCREASING SCALES OF MODULI. The scale discovered by Landen, and extended to the limits by Lagrange, implicitly assumes that $\sin \theta$ in one function F is proportional to the derivative of $\mathcal{A}$ in another function F ; that is, if $\tan \theta = \frac{\sin 2\theta'}{e + \cos 2\theta'}$, $\sin \theta = \frac{2}{1+e} \cdot \frac{\sin \theta' \cos \theta'}{\sqrt{[1-e'^2 \sin^2 \theta']}}$;
 $e' = \frac{2\sqrt{e}}{1+e}$. Or $\sin(2\theta' - \theta) = e \sin \theta$; $e'^2 = 1 - \left(\frac{1-e}{1+e}\right)^2$.
 Or $\cos(2\theta' - \theta) = \sqrt{[1-e^2 \sin^2 \theta]}$; $2 \div (1+e) = e' \div \sqrt{e}$.

The agreement of the fourth and sixth of these equations is proved by squaring and adding; and the agreement of the second equation with the first can be shown by squaring the first, then changing from $\sin \theta$ to $\cot \theta$, and eliminating $\cot \theta$, which leaves an identical equation.

Let us here introduce the auxiliary angle such that $\sin \varphi = e$, $\sin \varphi' = e'$, ... For computing the *ascending* scale;

$$e' = \frac{2\sqrt{e}}{1+e}; e'' = \frac{2\sqrt{e'}}{1+e'}; e''' = \frac{2\sqrt{e''}}{1+e''}; \dots$$

or $\tan \frac{1}{2}\varphi' = \sqrt{e} = \sqrt{(\sin \varphi)}$; $\tan \frac{1}{2}\varphi'' = \sqrt{(\sin \varphi')}$; $\tan \frac{1}{2}\varphi''' = \sqrt{(\sin \varphi'')}$; .. to be continued till the value of e is sufficiently near to the limit 1, which renders F an exact integral, as far as required.

For computing the series of amplitudes, we have

$$\begin{aligned} \sin(2\theta' - \theta) &= e \sin \theta = \sin \varphi \sin \theta; \\ \sin(2\theta'' - \theta') &= e' \sin \theta' = \sin \varphi' \sin \theta'; \\ \sin(2\theta''' - \theta'') &= e'' \sin \theta'' = \sin \varphi'' \sin \theta'' \dots \end{aligned}$$

Expanding $\sin(2\theta' - \theta)$, and dividing the equation by $\sin \theta$, we find

$$\cot \theta = \frac{e + \cos 2\theta'}{\sin 2\theta'}; \quad d \cot \theta = \frac{-(1+e \cos 2\theta') \cdot 2d\theta'}{\sin^2 2\theta'}.$$

$$\text{Now} \quad dF = \frac{d\theta}{\mathcal{A}} = \frac{-\sin^2 \theta \cdot d \cot \theta}{\mathcal{A}} = \frac{-\sin \theta \cdot d \cot \theta}{\sqrt{[1 + \cot^2 \theta - e^2]}}.$$

Substituting for $\sin \theta$ its value from the assumed equation above, and for $\cot \theta$ and $d \cot \theta$, the values just found, we have

$$dF = \frac{d\theta}{\sqrt{[1 - e^2 \sin^2 \theta]}} = \frac{2}{1+e} \cdot \frac{d\theta'}{\sqrt{[1 - e'^2 \sin^2 \theta']}}.$$

When the scale $e, e', e'', \dots$ has been extended so near to the limit 1 that the last modulus may be taken as 1, let θ denote the corresponding amplitude, which is also a limit; then

$$F(1, \theta) = \int \frac{d\theta}{\sqrt{[1 - \sin^2 \theta]}} = \int \frac{d\theta}{\cos \theta} = \log \tan(45^\circ + \frac{1}{2}\theta).$$

As the moduli increase, the amplitudes diminish to this limit; thus,

$$F(e, \theta) = \frac{2}{1+e} F(e', \theta') = \frac{2}{1+e} \cdot \frac{2}{1+e'} F(e'', \theta''), \dots$$

$$F(e, \theta) = \log \tan (45^\circ + \tfrac{1}{2}\theta) \cdot \frac{2}{1+e} \cdot \frac{2}{1+e'} \cdot \frac{2}{1+e''} \dots 1.$$

When $\theta = 90^\circ$, the *first* equation of amplitude reduces to $\sin (2\theta' - \theta) = 1.e = \sin \varphi$, or $\theta' = 45^\circ + \tfrac{1}{2}\varphi$.

$$\text{Let } M = \frac{2}{1+e} \cdot \frac{2}{1+e'} \cdot \frac{2}{1+e''} \dots 1 = \sqrt{\left(\frac{e'e''e''' \dots 1}{e} \right)} = \frac{\tan \tfrac{1}{2}\varphi'' \cdot \tan \varphi''' \dots 1}{\tan \tfrac{1}{2}\varphi'}.$$

$$F(e, \theta) = M \cdot \log \tan (45^\circ + \tfrac{1}{2}\theta).$$

The Napierian logarithm here indicated, is found by multiplying the common logarithm by 2.302585093.

Preparatory to the next section, the difference of the two equations, $\sin \theta = \sin \theta$, and $\sin (2\theta' - \theta) = e \sin \theta$, divided by their sum, gives $\tan (\theta - \theta') = [(1-e) \div (1+e)] \tan \theta' = \sqrt{[1-e^2]} \tan \theta'$.

Note. The relation discovered by Gauss in 1818 gives the same moduli as before, but a different series of amplitudes; thus, $e' = 2\sqrt{e} \div (1+e)$,

$$\sin \theta' = \frac{(1+e) \sin \theta}{1+e \sin^2 \theta}; \quad \frac{d\theta}{\sqrt{[1-e^2 \sin^2 \theta]}} = \frac{1}{1+e} \cdot \frac{d\theta'}{\sqrt{[1-e'^2 \sin^2 \theta']}}.$$

This scale of Gauss can be derived from that of Lagrange by changing from θ and θ' to Ψ and Ψ' after assuming $\cos \theta \cos \Psi = 1$, $\cos \theta' \cos \Psi' = 1$, as in Section II.

IV. DECREASING SCALES OF MODULI. The preceding scale of moduli can be prolonged in the opposite direction, tending to the limit 0; so that the entire scale will be 0, ... e^{000} , e^{00} , e^0 , e , e' , e'' , e''' , ... 1. Therefore reversing the accents in the above equation, we have for computing the decreasing series of moduli, as well as the amplitudes:

$$\frac{1-e^0}{1+e^0} = \sqrt{[1-e^2]} = b; \quad e^0 = \frac{1-b}{1+b}; \quad e^{00} = \frac{1-b^0}{1+b^0}; \quad e^{000} = \frac{1-b^{00}}{1+b^{00}}; \dots$$

$$e = \sin \varphi, \quad e^0 = \sin \varphi^0, \quad \dots; \quad e^0 = \tan^2 \tfrac{1}{2}\varphi; \quad e^{00} = \tan^2 \tfrac{1}{2}\varphi^0; \dots$$

$$e^0 = \sqrt{[1-b^{02}]} = \frac{1-b}{1+b}; \quad b^0 = \frac{2\sqrt{b}}{1+b}; \quad \frac{2}{1+b} = \frac{b^0}{\sqrt{b}}.$$

$$\tan(\theta^0 - \theta) = b \tan \theta = \cos \varphi \cdot \tan \theta,$$

$$\tan(\theta^{00} - \theta^0) = b^0 \tan \theta^0 = \cos \varphi^0 \cdot \tan \theta^0, \text{ etc.}$$

$$F(e, \theta) = \frac{1+e^0}{2} \cdot F(e^0, \theta^0) = \frac{1+e^0}{2} \cdot \frac{1+e^{00}}{2} \cdot F(e^{00}, \theta^{00}), \dots$$

(To be continued.)